Published by Institute of Physics Publishing for SISSA/ISAS

Received: *June 29, 2004*
Accepted: *July 8, 2004*

String excitation energies in $\mathrm{SU}(N)$ gauge theories beyond the free-string approximation

Martin Lüscher

CERN, Physics Department, TH Division
CH-1211 Geneva 23, Switzerland
E-mail: luscher@mail.cern.ch

Peter Weisz

Max-Planck-Institut für Physik
D-80805 Munich, Germany
E-mail: pew@mppmu.mpg.de

Abstract: In the presence of a static quark-antiquark pair, the spectrum of the low-lying states in $\mathrm{SU}(N)$ gauge theories is discrete and likely to be described, at large quark separations r, by an effective string theory. The expansion of the excitation energies in powers of $1/r$, which derives from the latter, involves an increasing number of unknown couplings that characterize the string self-interactions. Using open–closed string duality, we show that the possible values of the couplings are constrained by a set of algebraic relations. In particular, the corrections of order $1/r^2$ must vanish, while the $1/r^3$ terms (which we work out for the few lowest levels) depend on a single adjustable coupling only.

Keywords: Confinement, Lattice Gauge Field Theories, Bosonic Strings.

Contents

JHEP07(2004)014

1. Introduction

The formation of string-like flux tubes in $\mathrm{SU}(N)$ gauge theories is not a proved fact, but the results obtained in lattice gauge theory over the years leave little doubt that this physical picture is basically correct. Recent high-precision numerical simulations now even confirm the presence of characteristic string fluctuation effects in the static quark potential [1, 2] and in ratios of loop correlation functions [3]–[7].

A puzzling aspect of these findings is, however, that string behaviour appears to set in at quark-antiquark separations r as small as 0.5 to 1 fm or so, where the energy distribution in the ground state of the gauge field is still far from being tube-like. Moreover, at these distances the spectrum of the low-lying excited states is poorly matched by the regular pattern (with level splittings equal to π/r) that is obtained in the case of a free string with fixed ends [8, 9, 2]. Closed flux tubes wrapping around a compact space dimension have also been considered, with similar conclusions [10]–[14].

At present none of these results exclude that the observed mismatches will gradually disappear at larger distances r. The energy spectrum should in any case be compared with the predictions of the effective string theory of refs. [15]–[17], in which the string self-interactions are taken into account in a systematic manner. According to this theory, the free-string energies are only the leading terms in an asymptotic expansion in powers of $1/r$, while the higher-order corrections arise from interaction terms of increasing dimensions. There are finitely many terms at each order and a corresponding number of effective coupling constants.

Our principal goal in this paper is to work out the expansion of the few lowest energy levels up to order $1/r^3$. We shall first show, however, that the a priori unknown values of the couplings in the effective theory must satisfy certain algebraic relations in order to be consistent with the closed-string interpretation of the partition function. As a consequence, corrections proportional to $1/r^2$ are excluded and the number of independent couplings at the next order of the large-r expansion is reduced from 2 to 1. The predictive power of the effective theory is thus significantly enhanced.

2. Effective string theory

In this section we define the effective string theory and explain which properties of the gauge theory it is expected to describe. We also compute the string partition function to next-to-leading order, partly for illustration and partly because this will serve as a starting point for the discussion in the following section. Much of what is being said here already appeared in ref. [1], and we refer the reader to this paper for a less abbreviated presentation.

2.1 Definition

Eventually we are interested in the string energy spectrum in 4 space-time dimensions, but string formation is also studied in 3 dimensions and we therefore consider the effective theory in any dimension $d \geq 3$. The fluctuations of a string with fixed ends can be parametrized by a displacement vector $h(z)$, with $d-2$ components in the directions orthogoxnal to the string axis, where

$$z = (z_0, z_1), \qquad 0 \leq z_1 \leq r, \tag{2.1}$$

are the world-sheet coordinates. Fixed ends means that the field satisfies

$$h(z)|_{z_1=0} = h(z)|_{z_1=r} = 0 \tag{2.2}$$

for all times z_0, and in the time direction we shall impose periodic boundary conditions, with period T, at least as long as we are concerned with the string partition function.

The action of the effective theory is written in the form of a series

$$S = \sigma r T + \mu T + S_0 + S_1 + \cdots, \qquad S_0 = \frac{1}{2}\int \mathrm{d}^D z\,(\partial_a h \partial_a h), \tag{2.3}$$

in which σ denotes the string tension, μ an arbitrary mass parameter and $S_1, S_2, \ldots$ string self-interaction terms of increasing dimension. For the calculation of the string energy levels it is convenient to choose the metrics in space-time and on the world-sheet to be euclidean. We shall use dimensional regularization and thus extend the world-sheet to $D = 2 - \epsilon$ dimensions at this point (see appendix A).

As is generally the case in field theories on manifolds with boundaries [18, 19], the possible interaction terms are localized either on the world-sheet or at its boundaries. They may only depend on the derivatives of $h(z)$ and must respect the obvious symmetries of the system. Moreover, any terms that are formally removable by a field transformation can be dropped, since their effects on the energy spectrum amount to a renormalization of the coupling constants multiplying the interactions of lower dimension.

Inspection now shows that

$$S_1 = \frac{1}{4} b \int \mathrm{d}^{D-1} z\,\left\{(\partial_1 h \partial_1 h)_{z_1=0} + (\partial_1 h \partial_1 h)_{z_1=r}\right\} \tag{2.4}$$

is the only possible interaction term with coupling b of dimension [length]. At second order the couplings have dimension [length]^2 and the complete list of terms is

$$S_2 = \frac{1}{4} c_2 M^{\epsilon} \int \mathrm{d}^D z\,(\partial_a h \partial_a h)\,(\partial_b h \partial_b h), \tag{2.5}$$

$$S_3 = \frac{1}{4} c_3 M^{\epsilon} \int \mathrm{d}^D z\,(\partial_a h \partial_b h)\,(\partial_a h \partial_b h), \tag{2.6}$$

where M denotes the renormalization scale and the brackets indicate in which way the space indices are contracted. Since we are using dimensional regularization, there are no power divergences and b is thus not renormalized. In principle an additive renormalization of c_2, c_3 proportional to b^2 may be required, but it turns out that there are no logarithmic divergences to this order.

2.2 Partition function and the Polyakov loop correlation function

In the effective theory the partition function and the correlation functions of the field $h(z)$ can be expanded in powers of the couplings $b, c_2, c_3, \ldots$ following the usual steps. From this point of view the theory looks just like any other non-renormalizable quantum field theory. In particular, the perturbation series really is a low-energy expansion, where the higher-order terms are proportional to inverse powers of the external distances such as T and r.

To leading order the exact expression for the partition function

$$\mathcal{Z}_0 = \mathrm{e}^{-\sigma r T - \mu T} \eta(q)^{2-d}\,, \tag{2.7}$$

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n), \qquad q = \mathrm{e}^{-\pi T/r}\,, \tag{2.8}$$

is known since a long time (see refs. [20, 21, 3]] for example; the index "0" reminds us that we are dealing with the free-string theory). When expanded in powers of q, this formula leads to the representation

$$\mathcal{Z}_0 = \sum_{n=0}^{\infty} w_n \mathrm{e}^{-E_n^0 T}\,, \tag{2.9}$$

with some positive integral weights w_n and energies E_n^0 given by

$$E_n^0 = \sigma r + \mu + \frac{\pi}{r}\left\{-\frac{1}{24}(d-2) + n\right\}. \tag{2.10}$$

In particular, the splitting between successive energy levels is equal to π/r to this order of the perturbation expansion.

In the presence of a static quark-antiquark pair, the spectrum of the low-lying states in the (pure) SU(N) gauge theory is surely more complicated than this, but it is conceivable that the effective string theory provides an asymptotically correct description of the energy levels at large distances r. The partition function in this sector of the gauge theory is proportional to the Polyakov loop correlation function $\langle P(x)^* P(y)\rangle$ at distance $r = |\vec{x} - \vec{y}|$ on a euclidean space-time cylinder with circumference T in the time direction [1]. Evidently, if the energy spectra match (including multiplicities), the same must be true for the associated partition functions.

We would like to emphasize, however, that the (full) string partition function $\mathcal{Z}$ is only expected to match the Polyakov loop correlation function to any finite order of the perturbation expansion in the effective theory, i.e. only at large T and r and up to higher-order terms. Decays of highly excited states through glueball radiation are in fact not included in the string theory, at least not obviously so, and this must be reflected by relative differences in the partition functions of order e^{-mT}, where m denotes the mass of the lightest glueball.

2.3 Partition function at next-to-leading order

The order at which a given interaction term must first be included in the perturbation expansion coincides with the dimension of the associated coupling constant. At next-to-leading order the partition function thus reads

$$\mathcal{Z} = \mathcal{Z}_0 \left\{1 - \langle S_1 \rangle_0 \right\}, \tag{2.11}$$

where the expectation value $\langle \ldots \rangle_0$ is to be evaluated in the free-string theory. If we introduce the free propagator,

$$\langle h_k(z) h_l(w) \rangle_0 = \delta_{kl} G(z,w), \tag{2.12}$$

the first-order correction term assumes the form

$$\langle S_1 \rangle_0 = \frac{1}{4} b\,(d-2) \int \mathrm{d}^{D-1}z \,\left\{ \partial_1 G(z,z) \overleftarrow{\partial}_1 |_{z_1=0} + \partial_1 G(z,z) \overleftarrow{\partial}_1 |_{z_1=r} \right\} \tag{2.13}$$

and the rules of dimensional regularization must now be invoked to work it out.

Deferring the details of the calculation to appendix A, we note that there are no ultra-violet divergences and quote the result

$$\langle S_1 \rangle_0 = -b\,(d-2) \frac{\partial}{\partial r} \ln \eta(q). \tag{2.14}$$

The correction thus amounts to a shift $r \to r - b$ in the second factor of the leading-order expression (2.7) (this simple rule actually extends to the next order in b, but we shall not need to know this here).

3. Open-closed string duality and its consequences

We now proceed to show that the values of the coupling constants $b, c_2, c_3, \ldots$ in the effective string theory are constrained by certain algebraic relations if the Polyakov loop correlation function is to be matched by the string partition function at large T and r.

3.1 Dual interpretation of the Polyakov loop correlation function

Usually the Polyakov lines are considered to run along the time axis, forming closed loops through the periodic boundary conditions in this direction. This leads to the interpretation of the Polyakov loop correlation function as the partition function of the gauge theory in the presence of a static quark–antiquark pair.

We may, however, just as well take the compact dimension to be space-like, in which case the field-theoretical interpretation of the Polyakov loop $P(x)$ is that it creates (or annihilates) a closed flux tube wrapping around the world at, say, time x_1 and transverse spatial position $x_\perp = (x_2, \ldots, x_{d-1})$. The loop correlation function at zero transverse

JHEP07(2004)014

momentum is therefore expected to have a spectral representation of the form [22]–[25]

$$\int \mathrm{d}x_\perp \langle P(x)^* P(y) \rangle = \sum_{n=0}^{\infty} |v_n|^2 \, \mathrm{e}^{-\tilde{E}_n |x_1 - y_1|}, \tag{3.1}$$

where the exponents $\tilde{E}_n$ and the coefficients v_n are the energies and transition matrix elements of the possible intermediate states.[1] As shown in appendix B, this implies that the correlation function itself can be expanded in a series of Bessel functions according to

$$\langle P(x)^* P(y) \rangle = \sum_{n=0}^{\infty} |v_n|^2 \, 2r \left(\frac{\tilde{E}_n}{2\pi r} \right)^{\frac{1}{2}(d-1)} K_{\frac{1}{2}(d-3)}(\tilde{E}_n r) \,. \tag{3.2}$$

While the energy values and the transition matrix elements are not known, this equation (and similarly the spectral representation in the open-string channel) severely constrain the possible analytic form of the Polyakov loop correlation function.

3.2 Open-closed string duality at leading order

An interesting question is now whether the partition function in the effective string theory can be expanded, at large T and r, in a series of Bessel functions of the form (3.2). If not this would imply that the effective theory cannot possibly provide an asymptotically exact description of the Polyakov loop correlation function.

Using the modular transformation property of the η-function,

$$\eta(q) = \left(\frac{2r}{T} \right)^{\frac{1}{2}} \eta(\tilde{q}), \qquad \tilde{q} = \mathrm{e}^{-4\pi r/T} \,, \tag{3.3}$$

the leading-order expression (2.7) for the partition function can be written as a series of exponentials,

$$\mathcal{Z}_0 = \mathrm{e}^{-\mu T} \left(\frac{T}{2r} \right)^{\frac{1}{2}(d-2)} \sum_{n=0}^{\infty} w_n \mathrm{e}^{-\tilde{E}_n^0 r} \,, \tag{3.4}$$

where the weights w_n are the same as in the expansion (2.9) of the partition function in the open-string channel. Evidently, the closed-string energies

$$\tilde{E}_n^0 = \sigma T + \frac{4\pi}{T} \left\{ -\frac{1}{24}(d-2) + n \right\} \tag{3.5}$$

must coincide with the energies $\tilde{E}_n$ in eq. (3.2), up to terms of order b/T^2, if the series (3.2) and (3.4) are to match at large T and r.

Since their arguments grow proportionally to $\sigma T r$ in this limit, the Bessel functions in the series (3.2) may be expanded,

$$\langle P(x)^* P(y) \rangle = \sum_{n=0}^{\infty} |v_n|^2 \left(\frac{\tilde{E}_n}{2\pi r} \right)^{\frac{1}{2}(d-2)} \mathrm{e}^{-\tilde{E}_n r} \left\{ 1 + \frac{(d-2)(d-4)}{8\tilde{E}_n r} + \cdots \right\}, \tag{3.6}$$

[1] As in the case of the flux tubes with fixed ends, the higher excited states are unstable against glueball radiation and the spectrum of these states is thus continuous. This subtlety can be ignored in the following, since we shall be exclusively interested in the low energy values.

where, to leading order, we may replace $\tilde{E}_n$ by $\tilde{E}_n^0$ and drop all subleading terms in the curly bracket. The free-string partition function (3.4) is then seen to be of the required form, in any dimension d, and open-closed string duality thus holds to this order of the perturbation expansion.

3.3 Duality at next-to-leading order

The only interaction term that contributes at next-to-leading order is the boundary term S_1, and the partition function to this order is given by

$$\begin{aligned} \mathcal{Z} &= (1-\sigma T b)\,\mathcal{Z}_0 - b\frac{\partial}{\partial r}\mathcal{Z}_0 \\ &= \mathrm{e}^{-\mu T}\left(\frac{T}{2r}\right)^{\frac{1}{2}(d-2)}\sum_{n=0}^{\infty} w_n \left\{1 + b(\tilde{E}_n^0 - \sigma T) + \frac{b}{2r}(d-2)\right\}\mathrm{e}^{-\tilde{E}_n^0 r} \end{aligned} \tag{3.7}$$

(cf. subsection 2.3). From the discussion of the leading-order expression of the partition function we know that $\tilde{E}_n = \tilde{E}_n^0 + \mathrm{O}(b)$, and once this relation is inserted in eq. (3.6) it becomes clear that the b/r term in the curly bracket in eq. (3.7) cannot be matched. We thus conclude that

$$b = 0 \tag{3.8}$$

is the only possible value of this coupling, which is consistent with the requirement of open-closed string duality.

3.4 Duality at higher orders

At the next order of perturbation theory there are two string interaction terms, S_2 and S_3, with couplings c_2 and c_3. Their contribution to the partition function is discussed in section 4, where we shall find that open–closed string duality holds if and only if

$$(d-2)\,c_2 + c_3 = \frac{d-4}{2\sigma}\,. \tag{3.9}$$

The number of adjustable couplings is thus reduced to 1, and actually to 0 in dimension $d = 3$, since the interactions S_2 and S_3 are proportional to each other in this case (cf. section 6.2).

It is quite clear that the requirement of duality will lead to similar constraints at any order of perturbation theory. While this cannot be rigorously excluded, it seems unlikely that too many equations will be obtained at some point, because there are exact string partition functions with the correct properties in both the open and the closed string channels. The formal canonical quantization of the Nambu–Goto string, for example, yields such a partition function (appendix C).

4. Partition function at two-loop order

We now always set $b = 0$ so that the first non-trivial corrections to the free-string partition function come from the quartic interactions S_2 and S_3. At two-loop order the partition function is then given by

$$\mathcal{Z} = \mathcal{Z}_0\left\{1 - \langle S_2\rangle_0 - \langle S_3\rangle_0\right\}, \tag{4.1}$$

JHEP07(2004)014

where, as before, $\mathcal{Z}_0$ and the brackets $\langle \ldots \rangle_0$ stand for the partition function and the expectation values in the free theory.

4.1 Computation of $\langle S_2 \rangle_0$ and $\langle S_3 \rangle_0$

The contraction of the fields in the interaction terms S_2 and S_3 results in tadpole diagrams with two loops attached to the interaction vertex. We evaluated them using dimensional regularization, as described in appendix A, but the same diagrams were actually already calculated many years ago by Dietz and Filk [20] who employed a ζ-function finite-part prescription.

In terms of the series

$$\mathfrak{s}_1(q) = q \frac{\partial}{\partial q} \ln \eta(q) = \frac{1}{24} - \sum_{n=1}^{\infty} n q^n \left(1 - q^n\right)^{-1}, \tag{4.2}$$

$$\mathfrak{s}_2(q) = -q \frac{\partial}{\partial q} \mathfrak{s}_1(q) = \sum_{n=1}^{\infty} n^2 q^n \left(1 - q^n\right)^{-2} \tag{4.3}$$

(which are closely related to the first and second Eisenstein series), the result of the computation reads

$$\langle S_2 \rangle_0 = c_2 \, (d-2) \, \frac{\pi^2 T}{2r^3} \left\{ (d-1) \, \mathfrak{s}_2(q) + 2\mathfrak{s}_1(q)^2 \right\}, \tag{4.4}$$

$$\langle S_3 \rangle_0 = c_3 \, (d-2) \, \frac{\pi^2 T}{2r^3} \left\{ \frac{1}{2} (d+1) \, \mathfrak{s}_2(q) + (d-1) \, \mathfrak{s}_1(q)^2 \right\}. \tag{4.5}$$

In particular, the partition function is again a power series in $q = \mathrm{e}^{-\pi T/r}$, as in the case of the free-string theory.

4.2 Energy spectrum

Once the string interactions are switched on, the accidental degeneracies of the energy eigenstates, which are characteristic of the free-string theory, are not guaranteed to persist. The interpretation of the partition function as a spectral sum must take this possibility into account, and we thus write

$$\mathcal{Z} = \sum_{n=0}^{\infty} \sum_{i=1}^{k_n} w_{n,i} \, \mathrm{e}^{-E_{n,i} T}, \tag{4.6}$$

where k_n is the number of different energy values $E_{n,i}$ that are equal to E_n^0 to lowest order. Evidently the associated multiplicities $w_{n,i}$ must satisfy the sum rule

$$\sum_{i=1}^{k_n} w_{n,i} = w_n \tag{4.7}$$

for all $n \geq 0$.

It follows from this discussion that the partition function at two-loop order is of the general form

$$\mathcal{Z} = \sum_{n=0}^{\infty} w_n \left\{1 - [E_n^1]T\right\} \mathrm{e}^{-E_n^0 T} \,, \tag{4.8}$$

where $[E_n^1]$ is a linear combination of c_2 and c_3. This is just the weighted average, with weights equal to $w_{n,i}/w_n$, of the first-order energy shifts $E_{n,i}^1$ in the multiplet with principal quantum number n. In particular,

$$[E_0^1] = \left(\frac{\pi}{24}\right)^2 \frac{d-2}{2r^3} \left\{2c_2 + (d-1)\, c_3\right\} , \tag{4.9}$$

$$[E_1^1] = [E_0^1] + \frac{\pi^2}{24r^3} \left\{(12d-14)\, c_2 + (5d+7)\, c_3\right\} , \tag{4.10}$$

and the average energy shifts of the higher levels can be obtained straightforwardly by expanding the partition function in powers of q. Note that the degeneracies of the ground state and of the first excited states are unchanged by the interactions. There is thus a single level in these two cases, with first-order energy shift $E_{n,1}^1 = [E_n^1]$.

Using the fact that the series $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are derivatives of the free-string partition function, the result

$$[E_n^1] = [E_0^1] + \frac{\pi^2}{r^3} n \left[\frac{1}{12}(d-2) - n\right] c_2 \qquad \text{if} \quad c_3 = -2c_2 \,, \tag{4.11}$$

is now also easily established. Couplings related in this way are a rather special case in many respects. In particular, the techniques introduced in section 5 can be used to show that the free-string degeneracies of the energy levels are preserved by the interactions at this order of perturbation theory.

4.3 Closed-string representation

For simplicity we set $\mu = 0$ in the following, since the associated exponential factor only complicates the formulae, but is of no importance for the open-closed string duality of the partition function. The duality property of the free-string partition function may then be compactly expressed through

$$\mathcal{Z}_0 = \left(\frac{T}{2r}\right)^{\frac{1}{2}(d-2)} \mathcal{Z}_0|_{T\leftrightarrow 2r} \,, \tag{4.12}$$

where the notation on the right implies a substitution of T by $2r$ and simultaneously of r by $T/2$.

Similar identities can be derived for the first-order perturbations (4.4) and (4.5) since these are obtained from the η-function by differentiation. Explicitly one finds that

$$\begin{aligned} \langle S_2 + S_3\rangle_0 &= \langle S_2 + S_3\rangle_0|_{T\leftrightarrow 2r} + \\ &\quad + [(d-2)\, c_2 + c_3]\, (d-2)\, \frac{4\pi}{T^2} \left[\mathfrak{s}_1(\tilde{q}) - \frac{T}{16\pi r}\right] , \end{aligned} \tag{4.13}$$

and the closed-string representation of the partition function thus reads

$$\mathcal{Z} = \left(\frac{T}{2r}\right)^{\frac{1}{2}(d-2)} \Big\{ \mathcal{Z}|_{T\leftrightarrow 2r} + \\ + \left[(d-2)\,c_2 + c_3\right] \left[\frac{d-2}{4Tr} - \sigma - \frac{1}{T}\frac{\partial}{\partial r}\right] \mathcal{Z}_0|_{T\leftrightarrow 2r} \Big\} . \tag{4.14}$$

When expanded in powers of $\tilde{q}$, this formula yields the series

$$\mathcal{Z} = \sum_{n=0}^{\infty} w_n \left(\frac{T}{2r}\right)^{\frac{1}{2}(d-2)} \mathrm{e}^{-\tilde{E}_n^0 r} \Big\{ 1 - [\tilde{E}_n^1] r + \left[(d-2)\,c_2 + c_3\right] \times \\ \times \left[\frac{4\pi}{T^2}\left\{-\frac{1}{24}(d-2) + n\right\} + \frac{d-2}{4Tr}\right]\Big\} , \tag{4.15}$$

where the average closed-string energy shifts are given by

$$[\tilde{E}_n^1] = 2\,[E_n^1]_{r\to T/2} \, . \tag{4.16}$$

As in the case of the open-string channel, it is not possible, however, to deduce the degeneracy pattern of the energy levels from these equations.

The comparison with the right-hand side of eq. (3.6) now shows that open-closed string duality holds to this order of perturbation theory if and only if the couplings c_2 and c_3 are related to each other as in eq. (3.9). In $d = 4$ dimensions this implies, incidentally, that the couplings lie on the special line $c_3 = -2c_2$, where the accidental degeneracies of the energy levels in the open-string channel are not lifted by the interactions.

5. Computation of the energy shifts $E_{n,i}^1$

The first-order shifts of the individual energy levels can be extracted from the connected correlation functions of the field $h(z)$ at large time separations (we set $T = \infty$ in this section). Exactly which correlation functions are considered should not matter in principle, but the computations can be simplified by introducing fields that create and annihilate definite free-string modes. We now first explain this and then proceed with a discussion of the symmetry classification of the low-lying energy eigenstates and of the associated eigenvalue equation.

5.1 Creation and annihilation operators

Let us consider the fields

$$a_k^{\pm}(z_0, p) = \int_0^r \mathrm{d}z_1 \int_0^L \mathrm{d}z_2 \ldots \mathrm{d}z_{D-1}\, \sin(pz_1)\,(p \pm \partial_0)\, h_k(z) \, , \tag{5.1}$$

where $p = n\pi/r$, $n \in \mathbb{N}$, is an arbitrary spatial momentum and $k = 1, \ldots, d-2$ the space index that labels the directions transverse to the string axis. Since we intend to use dimensional regularization, the world-sheet is extended to D dimensions, as before, and an integration over the extra coordinates is included in eq. (5.1). It is straightforward to show that

$$\left\langle a_k^+(z_0,p)h_l(w)\right\rangle_0 = \delta_{kl}\theta(w_0-z_0)\,\mathrm{e}^{-p(w_0-z_0)}\sin(pw_1), \tag{5.2}$$

$$\left\langle a_k^-(z_0,p)h_l(w)\right\rangle_0 = \delta_{kl}\theta(z_0-w_0)\,\mathrm{e}^{-p(z_0-w_0)}\sin(pw_1), \tag{5.3}$$

which suggests that $a_k^+(z_0,p)$ and $a_k^-(z_0,p)$ create and annihilate harmonic string modes at time z_0 with wave number $n = pr/\pi$. The normalization of the associated one-mode states can be inferred from

$$\left\langle a_k^-(z_0,p)a_l^+(w_0,q)\right\rangle_0 = \delta_{kl}\delta_{pq}\theta(z_0-w_0)\,\mathrm{e}^{-p(z_0-w_0)}rpL^{-\epsilon}, \tag{5.4}$$

and we also note that

$$\left\langle a_k^+(z_0,p)a_l^+(w_0,q)\right\rangle_0 = \left\langle a_k^-(z_0,p)a_l^-(w_0,q)\right\rangle_0 = 0, \tag{5.5}$$

which is entirely in line with the interpretation of $a_k^+(z_0,p)$ and $a_k^-(z_0,p)$ as creation and annihilation operators (for simplicity equal-time contact terms were dropped in these equations).

Products of these fields create and annihilate states with any number of excited string modes. Effectively we have the association

$$a_{k_1}^+(z_0,p_1)\ldots a_{k_m}^+(z_0,p_m) \quad \longleftrightarrow \quad |n_1,\ldots,n_m;k_1,\ldots,k_m\rangle \tag{5.6}$$

between products of creation operators and the states in the free-string Fock space with transverse space-time indices $k_1,\ldots,k_m$ and wave numbers

$$n_j = \frac{p_j r}{\pi}, \qquad j = 1,\ldots,m. \tag{5.7}$$

Wick's rule ensures, incidentally, that the scalar products of these states are correctly given by the correlation functions of the corresponding products of the a^+ and a^- fields (the fields should be inserted at times infinitesimally different from each other so that contact terms do not arise).

5.2 Symmetry considerations

The Fock states (5.6) transform as tensors of rank m under the group $\mathrm{O}(d-2)$ of rotations about the string axis. Since the effective string theory is rotation-invariant, the energy eigenstates can be classified according to the irreducible representations of this group.

For general dimension d and principal quantum number $n = n_1 + \cdots + n_m \leq 3$, the Fock space decomposes into altogether 11 multiplets, as shown in table 1. The states listed in the table are Fock states, or linear combinations of these, that transform according to some irreducible representation of $\mathrm{O}(d-2)$. As usual the ground state is denoted by $|0\rangle$ and the states that are non-trivial combinations of Fock states are given

JHEP07(2004)014

n,i	State	$\mathrm{O}(d-2)$ multiplet	$w_{n,i}$
$0,1$	$\|0\rangle$	scalar	1
$1,1$	$\|1;k\rangle$	vector	$d-2$
$2,1$	$\|1,1\rangle$	scalar	1
$2,2$	$\|2;k\rangle$	vector	$d-2$
$2,3$	$\|1,1;\{k,l\}\rangle$	symmetric 2-tensor	$\frac{1}{2}d(d-3)$
$3,1$	$\|1,2\rangle$	scalar	1
$3,2$	$\|3;k\rangle$	vector	$d-2$
$3,3$	$\|1,1,1;k\rangle$	vector	$d-2$
$3,4$	$\|1,2;\{k,l\}\rangle$	symmetric 2-tensor	$\frac{1}{2}d(d-3)$
$3,5$	$\|1,2;[k,l]\rangle$	antisymmetric 2-tensor	$\frac{1}{2}(d-2)(d-3)$
$3,6$	$\|1,1,1;\{k,l,j\}\rangle$	symmetric 3-tensor	$\frac{1}{6}(d-2)(d+2)(d-3)$

Table 1: Energy eigenstates with quantum number $n \leq 3$.

by

$$|1,1\rangle = \sum_{k=1}^{d-2} |1,1;k,k\rangle\,, \tag{5.8}$$

$$|1,1;\{k,l\}\rangle = |1,1;k,l\rangle - \frac{\delta_{kl}}{d-2}\,|1,1\rangle\,, \tag{5.9}$$

$$|1,2\rangle = \sum_{k=1}^{d-2} |1,2;k,k\rangle\,, \tag{5.10}$$

$$|1,2;\{k,l\}\rangle = \frac{1}{2}\left\{|1,2;k,l\rangle + |1,2;l,k\rangle\right\} - \frac{\delta_{kl}}{d-2}\,|1,2\rangle\,, \tag{5.11}$$

$$|1,2;[k,l]\rangle = \frac{1}{2}\left\{|1,2;k,l\rangle - |1,2;l,k\rangle\right\}, \tag{5.12}$$

$$|1,1,1;k\rangle = \sum_{l=1}^{d-2} |1,1,1;k,l,l\rangle\,, \tag{5.13}$$

$$\begin{aligned} |1,1,1;\{k,l,j\}\rangle = {} & |1,1,1;k,l,j\rangle - \\ & - \frac{\delta_{kl}}{d}\,|1,1,1;j\rangle - \frac{\delta_{jk}}{d}\,|1,1,1;l\rangle - \frac{\delta_{lj}}{d}\,|1,1,1;k\rangle\,. \end{aligned} \tag{5.14}$$

It follows from this classification of states that they will not mix with each other when the interactions are switched on, except perhaps for the two vector multiplets at $n = 3$ that are indistinguishable from the symmetry point of view.

5.3 Eigenvalue equation

For specified quantum number n and representation of $\mathrm{O}(d-2)$, the states listed in table 1 span a subspace of lowest-order energy eigenstates with definite symmetry properties. The corresponding first-order energy shifts $E^1_{n,i}$ are obtained from the matrix elements of the interaction

$$S_{\mathrm{int}} = S_2 + S_3 \tag{5.15}$$

in these subspaces.

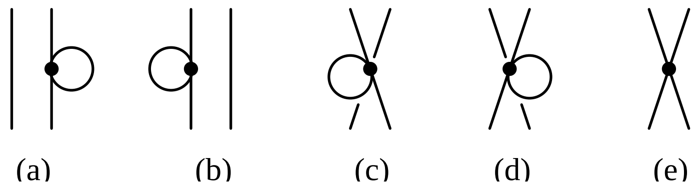

Figure 1: Feynman diagrams contributing to the four-point function (5.20). In this figure time runs vertically from time 0, where the string modes are created, up to time t, where they are annihilated.

Let $\phi_A^+(z_0)$ and $\phi_A^-(z_0)$ be the expressions in creation and annihilation operators at time z_0 that are associated to the basis vectors of a given subspace at level n. We label them by an arbitrary index A and define the normalization matrix N_{AB} through

$$\left\langle \phi_A^-(t)\phi_B^+(0)\right\rangle_0 = N_{AB}\,\mathrm{e}^{-n\pi t/r} \tag{5.16}$$

(we take t positive). The interaction matrix H_{AB} is similarly given by the correlation function

$$\left\langle \phi_A^-(t)S_{\mathrm{int}}\phi_B^+(0)\right\rangle_0 = \{H_{AB}t + \cdots\}\,\mathrm{e}^{-n\pi t/r}\,, \tag{5.17}$$

where the ellipsis stands for any terms independent of t.

From the spectral representation of the correlation functions in the interacting theory, it is now straightforward to deduce that the solutions λ of the generalized eigenvalue problem

$$Hv = \lambda N v \tag{5.18}$$

coincide with the first-order excitation energies

$$\Delta E^1_{n,i} = E^1_{n,i} - E^1_{0,1}\,. \tag{5.19}$$

If the representation of $\mathrm{O}(d-2)$ in the chosen subspace is irreducible, the eigenvalue problem is trivial, i.e. H is proportional to N and there is a single eigenvalue λ. The $n = 3$ vector multiplets are an exceptional case from this point of view, where one has two eigenvalues that correspond to $\Delta E^1_{3,2}$ and $\Delta E^1_{3,3}$.

5.4 Sample calculation

For illustration let us consider the correlation function

$$\left\langle a_k^-(t,p)a_l^-(t,q)S_{\mathrm{int}}a_i^+(0,p)a_j^+(0,q)\right\rangle_0 = \{M_{kl,ij}t + \cdots\}\,\mathrm{e}^{-(p+q)t}\,. \tag{5.20}$$

By a decomposition into irreducible $\mathrm{O}(d-2)$ tensors, the interaction matrices H_{AB} associated to the states in table 1 with two excited modes are then easily obtained from the matrix $M_{kl,ij}$. Here we only discuss the computation of this matrix, which amounts to the evaluation of the 5 diagrams shown in figure 1.

The diagrams 1(a)–(d) are products of the free propagator (5.4) and the connected part of the correlation function

$$\left\langle a_k^-(t,p)S_{\mathrm{int}}a_l^+(0,p)\right\rangle_0 = \{M_{kl}t + \cdots\}\,\mathrm{e}^{-pt}\,. \tag{5.21}$$

Using Wick's rule the combinatorial factors are easily obtained from this expression. Moreover, from eqs. (5.2) and (5.3) one infers that a non-zero contribution to the diagram is obtained only when the interaction vertex is inserted at any time between 0 and t. The part of the Feynman integral that adds to the matrix element M_{kl} is then easily identified, and its value may finally be calculated following the lines of appendix A. In the case of the diagram (a), for example, this leads to the result

$$M^{(a)}_{kl,ij} = \delta_{ki}\delta_{lj}\frac{pq^2}{4}\left\{\left[(d-1)\,qr - \frac{\pi}{6}\right]2c_2 + \left[(d+1)\,qr - \frac{\pi}{6}\,(d-1)\right]c_3\right\} \tag{5.22}$$

at $D = 2$.

To evaluate the tree diagram (e), we only need to substitute the free propagators (5.2) and (5.3) for the lines from the sources to the interaction vertex. It again follows from these formulae that the non-zero contributions are obtained when the vertex is at times between 0 and t. Some algebra then yields

$$\begin{aligned} M^{(e)}_{kl,ij} &= p^2q^2r\left\{\left[\delta_{ki}\delta_{lj}\delta_{pq} + \delta_{kj}\delta_{li} + \delta_{kl}\delta_{ij}\,(1+\delta_{pq})\right]c_2 + \right. \\ &\quad \left. + \left[\delta_{ki}\delta_{lj}\left(1+\frac{1}{2}\delta_{pq}\right) + \delta_{kj}\delta_{li}\left(\frac{1}{2}+\delta_{pq}\right) + \frac{1}{2}\delta_{kl}\delta_{ij}\,(1+\delta_{pq})\right]c_3\right\} \end{aligned} \tag{5.23}$$

for this diagram.

5.5 Why is $c_3 = -2c_2$ a special case?

If the couplings are related in this way, there are some important simplifications in the Feynman rules that allow the matrix elements H_{AB} to be calculated in closed form at any level n. We first observe that the interaction at $D = 2$ may be written in the form

$$S_{\text{int}} = -\frac{1}{4}c_2\int \mathrm{d}^2z\,(\partial_+h\partial_+h)\,(\partial_-h\partial_-h)\,, \qquad \partial_\pm = \partial_0 \pm i\partial_1\,, \tag{5.24}$$

if $c_3 = -2c_2$. In tree diagrams the source-to-vertex propagators are thus

$$\left\langle a^+_k(z_0,p)\partial_\pm h_l(w)\right\rangle_0 = \pm\delta_{kl}\theta(w_0 - z_0)ip\mathrm{e}^{-p(w_0-z_0)}\mathrm{e}^{\pm ipw_1}, \tag{5.25}$$

$$\left\langle a^-_k(z_0,p)\partial_\pm h_l(w)\right\rangle_0 = \pm\delta_{kl}\theta(z_0 - w_0)ip\mathrm{e}^{-p(z_0-w_0)}\mathrm{e}^{\mp ipw_1}\,, \tag{5.26}$$

and using these expressions it can be shown that the diagrams vanish, except when the states connected to the vertex are as in the diagram (e) considered above. Moreover, as can be seen from eq. (5.23), the contribution is then simply proportional to the identity matrix.

The only other diagrams are tadpole diagrams of the type 1(a)–(d), which are also proportional to the identity matrix. Combining all contributions it turns out that the interaction is diagonalized by the Fock states $|n_1,\ldots,n_m;k_1,\ldots,k_m\rangle$ and that the first-order energy shifts $E^1_n = [E^1_n]$ are given by eq. (4.11). Whether there is a deeper reason for this (a hidden symmetry perhaps) is, however, not known to us.

n,i	$\nu_{n,i}$	n,i	$\nu_{n,i}$
$1,1$	$\frac{1}{24}(5d+7)$	$3,2$	$\frac{1}{8}(17d+19)$
$2,1$	$\frac{1}{12}(11d+13)$	$3,3$	$\frac{1}{8}(9d+43)$
$2,2$	$\frac{1}{12}(11d+13)$	$3,4$	$\frac{1}{8}(9d+35)$
$2,3$	$\frac{1}{12}(5d+25)$	$3,5$	$\frac{1}{8}(9d+19)$
$3,1$	$\frac{1}{8}(17d+19)$	$3,6$	$\frac{1}{8}(5d+43)$

Table 2: Values of the coefficients $\nu_{n,i}$ in eq. (6.1).

6. Summary of results

6.1 Energy spectrum for general d, c_2 and c_3

As explained above, the obvious symmetries of the effective theory do not exclude that the $n = 3$ vector multiplets mix with each other once the interactions are switched on. The detailed calculation shows, however, that this does not happen to the order of perturbation theory considered. For $n \leq 3$ the first-order energy shifts are then all of the form

$$E^1_{n,i} = E^1_{0,1} + \frac{\pi^2}{r^3}\left\{n\left[\frac{1}{12}(d-2) - n\right]c_2 + \nu_{n,i}(c_3 + 2c_2)\right\}, \tag{6.1}$$

where the energy shift $E^1_{0,1} = [E^1_0]$ of the ground state is given by eq. (4.9) and the coefficients $\nu_{n,i}$ are listed in table 2.[2]

It follows from these results that most degeneracies of the energy levels are lifted by the interactions if $c_3 + 2c_2$ does not vanish. Equation (6.1) also shows that the first-order energy shifts tend to increase with n roughly like n^2. The distances r at which the large-r expansion of the energy values applies must therefore be expected to scale with n.

6.2 Energy spectrum in $d = 3$ dimensions

Since $h(z)$ has only one component in 3 dimensions, the interactions S_2 and S_3 are the same and all energy shifts $E^1_{n,i}$ are therefore proportional to $c_2 + c_3$. Without loss we may set $c_3 = -2c_2$, for example, which shows at once that this is a case where the free-string degeneracies of the energy levels are not lifted, i.e. to this order of perturbation theory there is a single energy value E_n for each n.

The duality constraint (3.9) now implies $c_2 = (2\sigma)^{-1}$ and the expansion

$$E_n = \sigma r + \mu + \frac{\pi}{r}\left(n - \frac{1}{24}\right) - \frac{\pi^2}{2\sigma r^3}\left(n - \frac{1}{24}\right)^2 + \mathrm{O}(r^{-4}) \tag{6.2}$$

of the energy values to this order thus involves no unknown parameters other than the string tension σ and the mass μ. Note that the "multiplets" listed in table 1 are either one-dimensional or do not occur in three dimensions.

[2]One readily checks that the average energy shifts $[E^1_n]$ obtained in section 4 from the partition function and the energy values quoted here satisfy the sum rule $\sum_i w_{n,i}E^1_{n,i} = w_n[E^1_n]$.

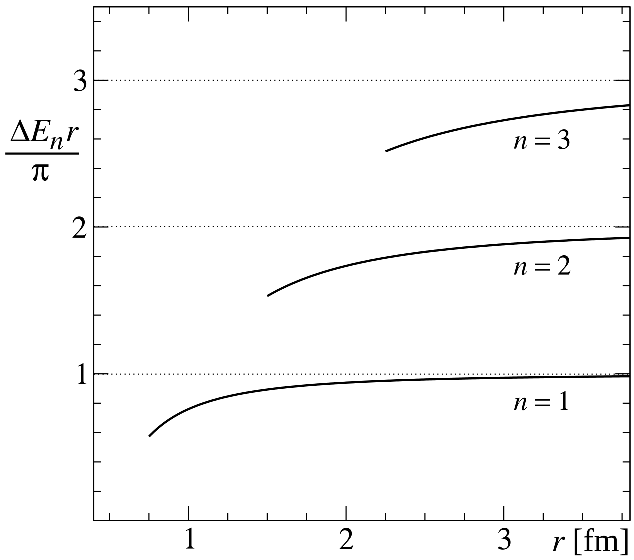

Figure 2: String excitation energies $\Delta E_n = E_n - E_0$ in four dimensions (eq. (6.3)) for coupling $c_2 = (2\sigma)^{-1}$. Physical units are defined by the Sommer scale $r_0 = 0.5$ fm and we have assumed that $r_0^2\sigma = 1.36$, as suggested by recent simulations of the SU(3) gauge theory.

6.3 Energy spectrum in $d = 4$ dimensions

Open-closed string duality requires $c_3 = -2c_2$ in this case, so that there is again only a single energy value E_n for each n. The coupling c_2 remains unconstrained, however, and the energies

$$E_n = \sigma r + \mu + \frac{\pi}{r}\left(n - \frac{1}{12}\right) - c_2 \frac{\pi^2}{r^3}\left(n - \frac{1}{12}\right)^2 + \mathrm{O}(r^{-4}) \tag{6.3}$$

depend on this coupling in a non-trivial way.

For illustration only, we plot the three lowest excitation energies in figure 2, in units of π/r and setting $c_2 = (2\sigma)^{-1}$. Taking this value as an example is suggested by our results in three dimensions and also by the fact that the coupling derived from the classical Nambu–Goto action has this value (the relation $c_3 = -2c_2$ holds in this theory too). The figure shows quite clearly that the corrections to the free-string energy splittings grow with n and that r must therefore be taken to increasingly large values to reach the asymptotic regime.

7. Concluding remarks

The calculations presented in this paper provide some insight into the effects of the string self-interactions on the energy spectrum in the effective string theory. In this approach the possible form of the interaction terms is constrained by symmetry considerations only, and the results are thus representative of a whole class of string theories rather than of any particular model.

JHEP07(2004)014

An important qualitative outcome in space-time dimensions $d \leq 4$ is that the free-string degeneracies of the energy levels are preserved to first order in the interactions. Since the number of interaction terms is rapidly increasing, it seems a bit unlikely, though, that the degeneracies will persist at higher orders of the perturbation expansion (unless the couplings assume some special values).

Our calculations also show that the first-order correction to the ground-state energy (the $1/r^3$ term) is significantly smaller than the corrections to the excited-state energies. Maybe this is why string behaviour in the static quark potential in SU(N) gauge theories is seen to set in at surprisingly short distances [1], where the energy spectrum is otherwise still far from being string-like [9]. However, the question that will have to be answered first is whether the low-lying energy values in the gauge theory are indeed matched by the string-theory formulae (6.2) and (6.3) at large distances r. The issue can presumably be resolved in lattice gauge theory, using numerical simulations, although this will require a dedicated effort and maybe more powerful simulation techniques to be able to reach the relevant range of distances with controlled systematic errors.

Acknowledgments

We are indebted to Hiroshi Suzuki and Wolfgang Lerche for their comments on the notion of open-closed string duality, and one of us (M.L.) wishes to thank the Yukawa Institute at Kyoto, where part of this paper was written, for the generous hospitality extended to him.

A. Dimensional regularization

The effective string theory lives on a manifold with boundary where the standard momentum-space techniques of dimensional regularization do not apply. There is a more general formulation of dimensional regularization, however, that can be used here since it is based on the heat-kernel representation of the propagator in position space [26].

A.1 Heat kernels

The coordinates of a point z on the world-sheet are $z_0, z_1, \ldots, z_{D-1}$, where the last $D-2$ parametrize the unphysical extra dimensions. We take these to be a product of circles, with circumference L, and now first need an explicit representation of the free massless propagator on this manifold, with Dirichlet boundary conditions at $z_1 = 0, r$ and periodic boundary conditions in all other directions.

To this end we introduce two types of one-dimensional heat kernels, the first

$$K_t^{\mathrm{P}}(x, y; T) = \frac{1}{T} \sum_{p_0} \mathrm{e}^{ip_0(x-y)} \mathrm{e}^{-tp_0^2}, \qquad p_0 = \frac{2\pi n_0}{T}, \quad n_0 \in \mathbb{Z}, \tag{A.1}$$

for periodic boundary conditions and the second

$$K_t^{\mathrm{D}}(x, y; r) = \frac{2}{r} \sum_{p_1} \sin(p_1 x) \sin(p_1 y) \mathrm{e}^{-tp_1^2}, \qquad p_1 = \frac{\pi n_1}{r}, \qquad n_1 \in \mathbb{N}, \tag{A.2}$$

for Dirichlet boundary conditions. The heat kernel and the propagator on the world-sheet are then given by

$$K_t(z,w) = K_t^{\mathrm{P}}(z_0,w_0;T)K_t^{\mathrm{D}}(z_1,w_1;r)\prod_{\alpha=2}^{D-1} K_t^{\mathrm{P}}(z_\alpha,w_\alpha;L)\,, \tag{A.3}$$

$$G(z,w) = \int_0^\infty \mathrm{d}t\, K_t(z,w)\,. \tag{A.4}$$

Evidently these formulae only make sense for non-negative integral values of $D-2$, but an analytic continuation in the dimension of the world-sheet is possible later, after the positions of the vertices in the Feynman diagrams have been integrated over.

Using the Poisson summation formula, the one-dimensional heat kernels can also be written in the form

$$K_t^{\mathrm{P}}(x,y;T) = (4\pi t)^{-1/2}\sum_{n=-\infty}^{\infty} \mathrm{e}^{-(x-y+nT)^2/4t}\,, \tag{A.5}$$

$$K_t^{\mathrm{D}}(x,y;r) = (4\pi t)^{-1/2}\sum_{n=-\infty}^{\infty} \left\{\mathrm{e}^{-(x-y+2nr)^2/4t} - \mathrm{e}^{-(x+y+2nr)^2/4t}\right\}. \tag{A.6}$$

In particular, in the limit $t\to 0$ the sums

$$\kappa_0(t) = \sum_{p_0} \mathrm{e}^{-tp_0^2}\,, \qquad \kappa_1(t) = \sum_{p_1} \mathrm{e}^{-tp_1^2}\,, \tag{A.7}$$

(where p_0 and p_1 run over the values specified above) are equal to $T(4\pi t)^{-1/2}$ and $-1/2 + r(4\pi t)^{-1/2}$ respectively, up to exponentially small terms.

A.2 Calculation of $\langle S_1\rangle_0$

Since the final results do not depend on the compactification scale L, we may now just as well set $L=T$ in this subsection. When the heat-kernel representation (A.4) of the propagator is inserted in eq. (2.13), the integral

$$\langle S_1\rangle_0 = -\frac{b}{r}\,(d-2)\int_0^\infty \mathrm{d}t\,\left[\kappa_0(t)\right]^{D-1}\frac{\mathrm{d}}{\mathrm{d}t}\kappa_1(t) \tag{A.8}$$

is obtained, which is absolutely convergent for all complex D in the range $\mathrm{Re}\,D<0$. Moreover, by adding and subtracting the asymptotic term $T(4\pi t)^{-1/2}$, raised to the power $D-1$, from the first factor in the integrand, the expression can be analytically continued in D to the whole complex plane. This leads to the representation

$$\langle S_1\rangle_0 = -\frac{b}{r}\,(d-2)\left\{\frac{\pi T}{24r} + \int_0^\infty \mathrm{d}t\,\left[\kappa_0(t) - T(4\pi t)^{-1/2}\right]\frac{\mathrm{d}}{\mathrm{d}t}\kappa_1(t)\right\}. \tag{A.9}$$

at $D=2$.

We may now expand $\kappa_0(t)$ and $\kappa_1(t)$ in the exponentials $\mathrm{e}^{-(mT)^2/4t}$ and $\mathrm{e}^{-(n\pi/r)^2 t}$ respectively and integrate the terms of the double-series over the parameter t. The sum over m can then be performed analytically, and the formula

$$\langle S_1\rangle_0 = -b\,(d-2)\,\frac{\pi T}{r^2}\left\{\frac{1}{24} - \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right\} \tag{A.10}$$

is thus obtained, which coincides with eq. (2.14).

A.3 More complicated diagrams

The one-particle irreducible parts of the Feynman diagrams considered in this paper are tadpole diagrams where one or more loops are attached to a single vertex. All these diagrams can be worked out analytically along the lines explained above.

In the first step the heat-kernel representation (A.4) is substituted for all (internal) propagators and the integral over the world-sheet coordinates of the interaction vertex is performed. One is then left with an integral over the heat-kernel parameters that is absolutely convergent for sufficiently small $\mathrm{Re}\, D$. The analytic continuation to the physical point $D = 2$ is finally achieved by separating the leading singularities of the integrand at small values of the heat-kernel parameters.

B. Radon transform of spherically symmetric functions

In this appendix we derive the expansion (3.2) from the spectral representation (3.1) and the fact that the Polyakov loop correlation function is spherically symmetric. The key point to note is that the left-hand side of eq. (3.1) has the form of an integral transformation that maps the correlation function to some function of $z = |x_1 - y_1|$. This is actually a special case of the Radon transform [27], and we now first discuss some general properties of this transformation.

B.1 Definition

In the following we consider smooth functions $f(r)$ on the half-line $r > 0$ that fall off faster than any inverse power of r at infinity. Their behaviour in the limit $r \to 0$ is left unspecified and can be arbitrarily singular.

For any integer $n \geq 2$ and all $z > 0$, we define the Radon transform $\mathfrak{R}_n f(z)$ of a given function $f(r)$ through

$$\mathfrak{R}_n f(z) = \int_{\mathbb{R}^{n-1}} \mathrm{d}^{n-1}y \; f(r)\big|_{r=\sqrt{y^2+z^2}} \,. \tag{B.1}$$

If we interpret r as the radius in $\mathbb{R}^n$, the integral on the right of this equation is taken over an $n-1$ dimensional hyperplane, parametrized by the coordinates $y \in \mathbb{R}^{n-1}$, at a distance z from the origin. It is easy to show that the transformed function is again in the class of functions specified above, and the transformation can thus be applied repeatedly. An immediate consequence of the definition (B.1) is then that

$$\mathfrak{R}_n \mathfrak{R}_m = \mathfrak{R}_{n+m-1} \,. \tag{B.2}$$

In particular, $\mathfrak{R}_n = \mathfrak{R}_2 \mathfrak{R}_2 \ldots \mathfrak{R}_2$ ($n-1$ factors).

B.2 Inversion of the Radon transform

In $n = 3$ dimensions we have

$$\mathfrak{R}_3 f(z) = 2\pi \int_z^\infty \mathrm{d}r \, r f(r) \,, \tag{B.3}$$

and the function $f(r)$ can be reconstructed from its Radon transform through

$$f(r) = -(2\pi r)^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\mathfrak{R}_3 f(r)\,. \tag{B.4}$$

The Radon transform $\mathfrak{R}_3$ is thus an invertible mapping from the space of functions defined above onto itself.

In view of the semi-group property (B.2), the same is also true for all other transformations. In particular, the inverse of $\mathfrak{R}_2$ is obtained by first acting with $\mathfrak{R}_2$ and then using eq. (B.4). Explicitly we have

$$\mathfrak{R}_n^{-1} = \begin{cases} (\mathfrak{R}_3^{-1})^{\frac{1}{2}(n-1)} & \text{if } n \text{ is odd,} \\ (\mathfrak{R}_3^{-1})^{\frac{1}{2}n}\mathfrak{R}_2 & \text{if } n \text{ is even,} \end{cases} \tag{B.5}$$

where $\mathfrak{R}_3^{-1} f(r) = -(2\pi r)^{-1} f'(r)$.

B.3 Example

It is now almost trivial to find the function $k(r)$ that satisfies

$$\mathfrak{R}_n k(z) = \mathrm{e}^{-Ez}\,. \tag{B.6}$$

If n is odd, say $n = 2\nu + 1$, it is given by

$$k(r) = (-2\pi)^{-\nu}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^{\nu}\mathrm{e}^{-Er} = 2r\left(\frac{E}{2\pi r}\right)^{\frac{n}{2}} K_{\frac{n}{2}-1}(Er)\,, \tag{B.7}$$

where the normalization of the Bessel functions is as in ref. [28]. In the other case the solution is obtained by noting that

$$\mathfrak{R}_2 f(z) = 2zK_1(Ez) \quad \text{if} \quad f(r) = \mathrm{e}^{-Er}\,. \tag{B.8}$$

The application of eq. (B.5) for $n = 2\nu$ then yields

$$k(r) = (-2\pi)^{-\nu}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^{\nu} 2rK_1(Er) = 2r\left(\frac{E}{2\pi r}\right)^{\frac{n}{2}} K_{\frac{n}{2}-1}(Er), \tag{B.9}$$

i.e. the same expression in terms of the Bessel functions as above.

B.4 Proof of eq. (3.2)

As already mentioned at the beginning of this appendix, eq. (3.1) may be written in the form

$$\mathfrak{R}_{d-1} f(z) = \sum_{n=0}^{\infty} |v_n|^2\,\mathrm{e}^{-\tilde{E}_n z}\,, \tag{B.10}$$

where $f(r)$ now stands for the Polyakov loop correlation function at distance r. The application of the inverse Radon transform and the results obtained in the previous subsection then immediately lead to the expansion (3.2).

A technical point we would like to emphasize here is that the inverse of the Radon transform is unique and that this crucial property holds irrespective of the behaviour of the functions at small distances r. The Radon transform is actually local in the sense that the values of the transformed function at $z \geq a$ only depend on the values of the input function at distances $r \geq a$, for any fixed $a > 0$.

JHEP07(2004)014

C. Open-closed string duality and the Nambu-Goto string

The energy spectrum of a Nambu-Goto string with fixed ends,

$$E_n = \sigma r \left\{ 1 + \frac{2\pi}{\sigma r^2} \left[-\frac{1}{24}(d-2) + n \right] \right\}^{1/2}, \qquad n = 0, 1, 2, \ldots, \tag{C.1}$$

was calculated some time ago by Arvis [29] through formal canonical quantization. Since the degeneracies of the energy eigenstates are the same as in the case of the free string, the associated partition function reads

$$\mathcal{Z} = \sum_{n=0}^{\infty} w_n \mathrm{e}^{-E_n T}. \tag{C.2}$$

We now derive the closed-string representation of this partition function.

Our starting point is the formula

$$\mathcal{Z} = \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{\mathrm{d}t}{t^{3/2}} \, \mathrm{e}^{-\frac{1}{4t} - (E_0 T)^2 t} \left\{ q^{-\frac{1}{24}} \eta(q) \right\}^{2-d}, \qquad q = \mathrm{e}^{-2\pi\sigma T^2 t}, \tag{C.3}$$

which is easily verified by expanding the curly bracket in powers of q and by integrating the resulting series term by term. The modular transformation property of the η-function may now be used and this leads, after re-expanding the curly bracket, to the representation

$$\mathcal{Z} = \frac{(\sigma T^2)^{\frac{1}{2}(d-2)}}{\sqrt{4\pi}} \int_0^\infty \mathrm{d}t \, t^{\frac{1}{2}(d-5)} \sum_{n=0}^{\infty} w_n \exp\left\{ -\frac{\tilde{E}_n^2}{4(\sigma T)^2 t} - (\sigma T r)^2 t \right\} \tag{C.4}$$

in which the closed-string energies are given by

$$\tilde{E}_n = \sigma T \left\{ 1 + \frac{8\pi}{\sigma T^2} \left[-\frac{1}{24}(d-2) + n \right] \right\}^{1/2}. \tag{C.5}$$

Integration of the terms in the sum (C.4) finally yields a series of Bessel functions of the form (3.2), with

$$|v_n|^2 = \frac{\sigma T}{\tilde{E}_n} \left(\frac{\pi}{\sigma} \right)^{\frac{1}{2}(d-2)} w_n, \tag{C.6}$$

and open-closed string duality thus holds exactly.

References

[1] M. Lüscher and P. Weisz, *Quark confinement and the bosonic string*, *J. High Energy Phys.* **07** (2002) 049 [hep-lat/0207003].

[2] P. Majumdar, *The string spectrum from large Wilson loops*, *Nucl. Phys.* **B 664** (2003) 213 [hep-lat/0211038].

[3] M. Caselle, R. Fiore, F. Gliozzi, M. Hasenbusch and P. Provero, *String effects in the Wilson loop: a high precision numerical test*, *Nucl. Phys.* **B 486** (1997) 245 [hep-lat/9609041].

[4] M. Caselle, M. Panero and P. Provero, *String effects in Polyakov loop correlators*, *J. High Energy Phys.* **06** (2002) 061 [hep-lat/0205008].

JHEP07(2004)014

[5] M. Caselle, M. Hasenbusch and M. Panero, *String effects in the 3D gauge ising model*, *J. High Energy Phys.* **01** (2003) 057 [hep-lat/0211012]; *Effective string picture for confinement at finite temperature: theoretical predictions and high precision numerical results*, *Nucl. Phys.* **129** *(Proc. Suppl.)* (2004) 593 [hep-lat/0309147].

[6] M. Caselle, M. Pepe and A. Rago, *String effects in* SU(2) *lattice gauge theory*, *Nucl. Phys.* **129** *(Proc. Suppl.)* (2004) 721 [hep-lat/0310005]; *Static quark potential and effective string corrections in the (2+1)-d* SU(2) *Yang-Mills theory*, hep-lat/0406008.

[7] M. Caselle, M. Hasenbusch and M. Panero, *Short distance behaviour of the effective string*, *J. High Energy Phys.* **05** (2004) 032 [hep-lat/0403004].

[8] S. Perantonis and C. Michael, *Static potentials and hybrid mesons from pure* SU(3) *lattice gauge theory*, *Nucl. Phys.* **B 347** (1990) 854.

[9] K.J. Juge, J. Kuti and C. Morningstar, *Gluon excitations of the static quark potential and the hybrid quarkonium spectrum*, *Nucl. Phys.* **63** *(Proc. Suppl.)* (1998) 326 [hep-lat/9709131]; *Where is the string limit in QCD?*, *Nucl. Phys.* **73** *(Proc. Suppl.)* (1999) 590 [hep-lat/9809098]; *Quark confinement and surface critical phenomena*, *Nucl. Phys.* **83** *(Proc. Suppl.)* (2000) 503 [hep-lat/9911007]; *The QCD string spectrum and conformal field theory*, *Nucl. Phys.* **106** *(Proc. Suppl.)* (2002) 691 [hep-lat/0110157]; *Fine structure of the QCD string spectrum*, *Phys. Rev. Lett.* **90** (2003) 161601 [hep-lat/0207004].

[10] G. Parisi, R. Petronzio and F. Rapuano, *A measurement of the string tension near the continuum limit*, *Phys. Lett.* **B 128** (1983) 418.

[11] P. de Forcrand, G. Schierholz, H. Schneider and M. Teper, *The string and its tension in* SU(3) *lattice gauge theory: towards definitive results*, *Phys. Lett.* **B 160** (1985) 137.

[12] S. Ohta, M. Fukugita and A. Ukawa, *Forces between quark clusters*, *Phys. Lett.* **B 173** (1986) 15.

[13] B. Lucini and M. Teper, *Confining strings in* SU(N) *gauge theories*, *Phys. Rev.* **D 64** (2001) 105019 [hep-lat/0107007].

[14] K.J. Juge, J. Kuti, F. Maresca, C. Morningstar and M.J. Peardon, *Excitations of torelon*, *Nucl. Phys.* **129** *(Proc. Suppl.)* (2004) 703 [hep-lat/0309180].

[15] Y. Nambu, *QCD and the string model*, *Phys. Lett.* **B 80** (1979) 372.

[16] M. Lüscher, K. Symanzik and P. Weisz, *Anomalies of the free loop wave equation in the WKB approximation*, *Nucl. Phys.* **B 173** (1980) 365.

[17] M. Lüscher, *Symmetry breaking aspects of the roughening transition in gauge theories*, *Nucl. Phys.* **B 180** (1981) 317.

[18] K. Symanzik, *Schrödinger representation and Casimir effect in renormalizable quantum field theory*, *Nucl. Phys.* **B 190** (1981) 1.

[19] M. Lüscher, *Schrödinger representation in quantum field theory*, *Nucl. Phys.* **B 254** (1985) 52.

[20] K. Dietz and T. Filk, *On the renormalization of string functionals*, *Phys. Rev.* **D 27** (1983) 2944.

[21] J. Ambjørn, P. Olesen and C. Peterson, *Observation of a string in three-dimensional* SU(2) *lattice gauge theory*, *Phys. Lett.* **B 142** (1984) 410; *Three-dimensional lattice gauge theory and strings*, *Nucl. Phys.* **B 244** (1984) 262.

[22] M. Lüscher, *Construction of a selfadjoint, strictly positive transfer matrix for euclidean lattice gauge theories*, *Commun. Math. Phys.* **54** (1977) 283.

[23] K. Osterwalder and E. Seiler, *Gauge field theories on the lattice*, *Ann. Phys. (NY)* **110** (1978) 440.

[24] E. Seiler, *Gauge theories as a problem of constructive quantum field theory and statistical mechanics*, Lecture Notes in Physics 159 Springer, Berlin, 1982.

[25] M. Lüscher, *Selected topics in lattice field theory*, Lectures given at Les Houches 1988, in *Fields, strings and critical phenomena*, E. Brézin, J. Zinn-Justin eds., North-Holland, Amsterdam, 1989.

[26] M. Lüscher, *Dimensional regularization in the presence of large background fields*, *Ann. Phys. (NY)* **142** (1982) 359.

[27] I.M. Gelfand, M.I. Graev, N.Y. Vilenkin, *Generalized functions*, Academic Press, New York, 1966,vol. 5.

[28] I.S. Gradshteyn, I.M. Ryzhik, *Table of integrals, series and products*, Academic Press, New York, 1965.

[29] J.F. Arvis, *The exact $q\bar{q}$ potential in Nambu string theory*, *Phys. Lett.* **B 127** (1983) 106.